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SYMMETRIC SPACES WHICH ARE REAL COHOMOLOGY SPHERES

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This is a survey in which we collate some known results using semi-standard techniques, dropping the condition of simple connectivity in Kostant's work [2] and proving

Theorem 1. *Let M be a compact connected riemannian symmetric space. Then M is a real cohomology* ($\dim M$)*-sphere if and only if*

(1) *M is an odd dimensional sphere or real projective space; or*

(2) $M = \bar{M}/\Gamma$ *where* (a) $\bar{M} = \mathbf{S}^{2r_1} \times \cdots \times \mathbf{S}^{2r_m}$, $r_i > 0$, *product of $m \geq 1$ even dimensional spheres, and* (b) *Γ consists of all $\gamma = \gamma_1 \times \cdots \times \gamma_m$, where γ_i is the identity map or the antipodal map of $\mathbf{S}^{2r_i}$, and the number of γ_i which are antipodal maps, is even; or*

(3) $M = \mathbf{SU}(3)/\mathbf{SO}(3)$ *or* $M = \{\mathbf{SU}(3)/\mathbf{Z}_3\}/\mathbf{SO}(3)$; *or*

(4) $M = \mathbf{O}(5)/\mathbf{O}(2) \times \mathbf{O}(3)$, *non-oriented real grassmannian of 2-planes through 0 in $\mathbf{R}^5$.*

In (2) we note $\pi_1(M) = \Gamma \cong (\mathbf{Z}_2)^{m-1}$; in particular the even dimensional spheres are the case $m = 1$. In (3) we note that the first case is the universal 3-fold covering of the second case. In (4) we have $\pi_1(M) \cong \mathbf{Z}_2$.

Theorem 1 is based on a series of lemmas which can be pushed, with appropriate modification, to the case of a real cohomology n-sphere of dimension greater than n. *Here we make the convention that a 0-sphere is a single point.* By using a cohomology theory which satisfies the homotopy axiom (such as singular theory) we can also drop the requirement of compactness. Thus we push the method of proof of Theorem 1 and obtain

Theorem 2. *Let M be a connected riemannian symmetric space. Then M is a real cohomology n-sphere, $0 \leq n \leq \dim M$, if and only if $M = M' \times M''$ where (α) M'' is a product whose $l \geq 0$ factors are euclidean spaces and irreducible symmetric spaces of noncompact type, and (β) M' is one of the following spaces.*

(1) $M' = \bar{M}/\Gamma^\theta$, *where $\bar{M} = \mathbf{S}^{2r_1} \times \cdots \times \mathbf{S}^{2r_m}$ is the product of $m \geq 0$ spheres of positive even dimensions $2r_i$, $\Gamma \cong (\mathbf{Z}_2)^m$ consists of all $\gamma_1 \times \cdots \times \gamma_m$ such that γ_i is the identity or antipodal map on $\mathbf{S}^{2r_i}$, θ is any one of the 2^m characters on Γ, and Γ^θ is the kernel of θ. Express $\theta = \theta_{i_1} \cdots \theta_{i_s}$, where*

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$1 \leq i_1 < \cdots < i_s \leq m$, *and* θ_i *is the nontrivial character on the* $\mathbf{Z}_2$*-factor of* Γ *for* $\mathbf{S}^{2r_i}$. *Then* $n = 2r_{i_1} + \cdots + 2r_{i_m}$; *so either* $\theta = 1$ *with* $n = 0$ *and* $\Gamma^\theta = \Gamma \cong (\mathbf{Z}_2)^m$, *or* $\theta \neq 1$ *with* $n > 0$ *and* $\Gamma^\theta \cong (\mathbf{Z}_2)^{m-1}$.

(2a) $M' = (\mathbf{S}^{2r+1}/\mathbf{Z}_2) \times (\overline{M}/\Gamma)$, $r \geq 1$, *and* $\overline{M}$ *and* Γ *as in* (1), *product of an odd dimensional real projective space with* $m \geq 0$ *even dimensional real projective spaces*; $n = 2r + 1$.

(2b) $M' = (\mathbf{S}^{2r+1} \times \overline{M})/\Gamma_\theta$, $r \geq 0$, *and* $\overline{M}$ *and* Γ *as in* (1), *where* θ *is any of the* 2^m *characters on* Γ *(viewed as taking values in the group* $\mathbf{Z}_2$ *consisting of* 1 *and the antipodal map of* $\mathbf{S}^{2r+1}$*), and* Γ_θ *consists of all* $\theta(\gamma) \times \gamma$ *with* $\gamma \in \Gamma$; $n = 2r + 1$ *and* $\Gamma_\theta \cong (\mathbf{Z}_2)^m$.

(3) $M' = (\{\mathbf{SU}(3)/\mathbf{SO}(3)\} \times \overline{M})/\Psi$, *where* $\overline{M}$ *and* Γ *are as in* (1), $\mathbf{Z}_3$ *is the center of* $\mathbf{SU}(3)$, *and either* $\Psi = \{1\} \times \Gamma \cong (\mathbf{Z}_2)^m$ *or* $\Psi = \mathbf{Z}_3 \times \Gamma \cong \mathbf{Z}_3 \times (\mathbf{Z}_2)^m$; $n = 5$.

(4) $M' = (\{\mathbf{SO}(5)/\mathbf{SO}(2) \times \mathbf{SO}(3)\}/\mathbf{Z}_2) \times (\overline{M}/\Gamma)$, *where* $\overline{M}$ *and* Γ *are as in* (1); *the first factor of* M' *is the non-oriented grassmannian of* 2*-planes in* $\mathbf{R}^5$, *expressed as quotient of the oriented grassmannian by* $\{1, \eta\} = \mathbf{Z}_2$, *where* η *changes the orientation of each* 2*-plane*; $n = 6$.

(5) $M' = (\{\mathbf{SO}(6)/\mathbf{SO}(3) \times \mathbf{SO}(3)\}/\mathbf{Z}_4) \times (\overline{M}/\Gamma)$, *where* $\overline{M}$ *and* Γ *are as in* (1); *the first factor of* M' *is quotient of the oriented grassmannian of* 3*-planes in* $\mathbf{R}^6$ *by* $\{1, \beta, \beta^2, \beta^3\} = \mathbf{Z}_4$, *where* β *is oriented orthocomplementation of* 3*-planes so* $\beta^2 = \eta$ *orientation change*; $n = 5$.

As an immediate consequence of Theorem 2, or of Theorem 1 in the case $n = \dim M$ to which it applies, we have

Corollary. *Let* M *be a connected riemannian symmetric space which is a real cohomology* n*-sphere. If a prime* $p > 3$, *then* M *is a* $\mathbf{Z}_p$*-cohomology* n*-sphere.* M *is an integral cohomology* n*-sphere if and only if* $M = \mathbf{S}^n \times M''$ *with* M'' *acyclic as in condition* (α) *of Theorem* 2.

1. Cohomology invariants of deck transformations

Let M be a compact connected riemannian symmetric space. Let $\mathbf{I}(M)$ denote the full group of isometries of M, and $\mathbf{I}_0(M)$ its identity component. Now $M = G/K$, where $G = \mathbf{I}_0(M)$, compact connected Lie group, and K is the isotropy subgroup at a point $x \in M$. Let $s \in \mathbf{I}(M)$ denote the symmetry at x. Then the Lie algebra of G decomposes as $\mathbf{G} = \mathbf{K} + \mathbf{P}$ into (± 1)-eigenspaces of $ad(s)$, $\mathbf{K}$ being the subalgebra of $\mathbf{G}$ for K and $\mathbf{P}$ representing the tangent space of M at x. Using de Rham's Theorem and then averaging differential forms over G, one obtains a graded algebra isomorphism of $H^*(M; \mathbf{R})$ onto the space of $ad_G(K)$-invariant elements of $\Lambda^*\mathbf{P}' = \Sigma\Lambda^k\mathbf{P}'$ where $'$ denotes dual space. That is É. Cartans's representation of cohomology by invariant differential forms; an exposition is given in [4, § 8.5].

In particular, M is a real cohomology $(\dim M)$-sphere if and only if the only $ad_G(K)$-invariants in $\Lambda^*\mathbf{P}'$ are the linear combinations of $1 \in \Lambda^0\mathbf{P}'$ and the volume element $\omega \in \Lambda^n\mathbf{P}'$, $n = \dim M$.

M has universal riemannian covering $\varphi: N \to M$, where $N = N_0 \times M_1 \times \cdots \times M_r$, N_0 is a euclidean space, and the M_i are compact simply connected irreducible riemannian symmetric spaces. Let $\Delta \subset \mathbf{I}(N)$ be the group of deck transformations, so $M = N/\Delta$. Then $\Delta_0 = \Delta \cap \mathbf{I}(N_0)$ is a translation lattice on N_0, so $M_0 = N_0/\Delta_0$ is a flat riemannian torus, and φ factors through $\pi: \overline{M} \to M = \overline{M}/\Gamma$ where

$$\overline{M} = M_0 \times M_1 \times \cdots \times M_r , \qquad \Gamma = \Delta/\Delta_0 .$$

Let $\overline{G} = \mathbf{I}_0(\overline{M})$, $\bar{x} \in \pi^{-1}(x)$, and $\overline{K}$ be the isotropy subgroup of $\overline{G}$ at $\bar{x}$. Then we have an identification of $\overline{\mathbf{G}}$ with $\mathbf{G}$, which matches $\overline{\mathbf{K}}$ with $\mathbf{K}$ and $\overline{\mathbf{P}}$ with $\mathbf{P}$.

1.1. Lemma. *Identify $H^*(\overline{M}; \mathbf{R})$ with the $ad_{\overline{G}}(\overline{K})$-invariants on $\Lambda^*\mathbf{P}'$, and $H^*(M; \mathbf{R})$ with the $ad_G(K)$-invariants on $\Lambda^*\mathbf{P}'$. Then $H^*(M; \mathbf{R})$ consists of the Γ-invariants on $H^*(\overline{M}; \mathbf{R})$.*

For $G = (\overline{G}\Gamma)/\Gamma$ and $K = (\overline{K}\Gamma)/\Gamma$, and the cohomology of M is given by G-invariant differential forms.

Let $G_i = \mathbf{I}_0(M_i)$, and let Z_i denote the centralizer of G_i in $\mathbf{I}(M_i)$. Then $Z_0 = G_0$, the other Z_i are finite, $\overline{G} = G_0 \times G_1 \times \cdots \times G_r$, and $\overline{Z} = Z_0 \times Z_1 \times \cdots \times Z_r$ is the centralizer of $\overline{G}$ in $\mathbf{I}(\overline{M})$. Given a subgroup $\Psi \subset \mathbf{I}(\overline{M})$, one knows that $\overline{M} \to \overline{M}/\Psi$ is a riemannian covering with symmetric quotient, if and only if Ψ is a finite subgroup of $\overline{Z}$. Thus $\Gamma \subset \overline{Z}$. We write Γ_i for the projection of Γ to Z_i.

1.2. Lemma. *Let M be a real cohomology n-sphere, $n =$ $\dim M$. Then we have just one of the following situations.*

(a) *M is a circle.*

(b) *$\overline{M}$ is irreducible, the Γ-invariants on $H^*(\overline{M}; \mathbf{R})$ are generated by* 1 *and the volume element, and the Z-invariants on $H^*(M; \mathbf{R})$ are generated either by* 1 *or by* 1 *and the volume element.*

(c) *$\overline{M} = M_1 \times \cdots \times M_r$ with $r > 1$; for each i,* $\dim M_i > 0$ *and the Z_i-invariants on $H^*(M_i; \mathbf{R})$ are just the elements $1 \cdot \mathbf{R}$ of degree* 0.

Proof. Suppose $\dim M_0 > 0$. As Z_0 acts trivially on $H^*(M_0; \mathbf{R})$ it follows that $H^*(M; \mathbf{R})$ has nonzero elements of degree $\dim M_0$. Thus M is the torus M_0. Now $\dim M_0 = 1$, so M is a circle and we are in case (a).

If $\overline{M}$ is irreducible, the part of (b) on Γ-invariants is obvious and the statement on Z-invariants follows.

Now suppose that we are neither in case (a) nor in case (b). Then $\dim M_0 = 0$ and $\overline{M}$ is reducible, so $\overline{M} = M_1 \times \cdots \times M_r$ with $r > 1$ and $\dim M_i > 0$. If ϕ is a Z_i-invariant of positive degree on $H^*(M_i; \mathbf{R})$, then ϕ is Γ-invariant, so $\phi \in H^*(M; \mathbf{R})$ with $0 < \deg \phi < \dim M$. Thus the Z_i-invariants on $H^*(M_i; \mathbf{R})$ are of degree 0.

2. Admissible factors of M

We go on to find the irreducible symmetric spaces which satisfy the conditions imposed by (b) or (c) of Lemma 1.2.

2.1. Proposition. *Let M be a compact irreducible simply connected riemannian symmetric space, $G = \mathbf{I}_0(M)$, and Z be the centralizer of G in $\mathbf{I}(M)$. Then the Z-invariants on $H^*(M; \mathbf{R})$*

(i) *are all of degree 0, if and only if M is an even dimensional sphere;*

(ii) *are generated by 1 and the volume element, if and only if M is an odd dimensional sphere, $\mathbf{SU}(3)/\mathbf{SO}(3)$, or $\mathbf{SO}(5)/\mathbf{SO}(2) \times \mathbf{SO}(3)$.*

Proof. If M is a sphere the assertion is clear. If M is a real cohomology sphere but not a sphere, then [2] $M = \mathbf{SU}(3)/\mathbf{SO}(3)$, so [4, § 9.6] Z is the center of G and the assertion follows. Now suppose that M is not a real cohomology sphere. Then Z acts nontrivially on $H^*(M; \mathbf{R})$, so $Z \not\subset G$. It follows [4, Chapters 8 and 9], [3, § 5] that M is one of the spaces:

(1) $M = \mathbf{SU}(2n)/\mathbf{S}[\mathbf{U}(n) \times \mathbf{U}(n)]$, $Z \cong \mathbf{Z}_2$;

(2) $M = \mathbf{SU}(2n)/\mathbf{SO}(2n)$, $Z \cong \mathbf{Z}_{2n}$;

(3) $M = \mathbf{SO}(2r + 2s + 1)/\mathbf{SO}(2r) \times \mathbf{SO}(2s + 1)$, $Z \cong \mathbf{Z}_2$;

(4) $M = \mathbf{SO}(4n)/\mathbf{U}(2n)$, $Z \cong \mathbf{Z}_2$;

(5) $M = \mathbf{SO}(2r + 2s)/\mathbf{SO}(2r) \times \mathbf{SO}(2s)$,
$Z \cong \mathbf{Z}_2$ if $r \neq s$, $Z \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ if $r = s$;

(6) $M = \mathbf{SO}(4r + 2)/\mathbf{SO}(2r + 1) \times \mathbf{SO}(2r + 1)$, $Z \cong \mathbf{Z}_4$;

(7) $M = \mathbf{Sp}(n)/\mathbf{U}(n)$, $Z \cong \mathbf{Z}_2$;

(8) $M = \mathbf{Sp}(2n)/\mathbf{Sp}(n) \times \mathbf{Sp}(n)$, $Z \cong \mathbf{Z}_2$;

(9) $M = \mathbf{E}_7/\mathbf{A}_7$, $Z \cong \mathbf{Z}_2$;

(10) $M = \mathbf{E}_7/\mathbf{E}_6\mathbf{T}_1$, $Z \cong \mathbf{Z}_2$.

Let $M = G/K$ be one of the spaces above. If $\operatorname{rank} G = \operatorname{rank} K$, i.e. if the Euler-Poincaré characteristic $\chi(M) \neq 0$, then we have $\chi(M) = |W_G| / |W_K|$ where W_L = Weyl group of L. As cohomology occurs only in even degree, and as $\chi(M/Z) = \chi(M)/|Z|$, it follows that the two conditions for Z-invariants on $H^*(M; \mathbf{R})$ can be phrased

(i) $\chi(M/Z) = 1$, i.e. $|W_G| / |W_K| \cdot |Z| = 1$;

(ii) $\chi(M/Z) = 2$, i.e. $|W_G| / |W_K| \cdot |Z| = 2$.

We run through the relevant cases.

(1) $\chi(M/Z) = (2n)!/n!n!2$ which is > 2 whenever $n > 1$; we exclude $n = 1$ by the condition that M is not a sphere $\mathbf{S}^2$.

(3) $r \geq 1$ because $\dim M > 0$, and $s \geq 1$ because M is not a sphere. Thus $t = \min(r, s) \geq 1$. Now

$$\chi(M/Z) = 2^{r+s}(r + s)!/\{2^{r-1}r!\}\{2^s s!\}2 = (r + s)!/r!s! \geq (2t)!/t!t!$$

with equality if and only if $r = s$, and $(2t)!/t!t! \geq 2$ with equality if and only if $t = 1$. Thus $r = s = 1$, so $M = \mathbf{SO}(5)/\mathbf{SO}(2) \times \mathbf{SO}(3)$, and $\chi(M/Z) = 2$.

(4) $\chi(M/Z) = 2^{2n-1}(2n)!/(2n)!2 = 2^{2n-2}$ which is > 2 whenever $n > 1$; we exclude $n = 1$ because M is not a product $\mathbf{S}^2 \times \mathbf{S}^2$ of spheres.

(5) $r \geq 1$ and $s \geq 1$ because $\dim M > 0$. We exclude the case $r = s = 1$ because M is not a product $\mathbf{S}^2 \times \mathbf{S}^2$ of spheres. Now we may assume $1 \leq r \leq s$ with $s > 1$. If $r = s$, then

$$\chi(M/Z) = 2^{2r-1}(2r)!/\{2^{r-1}r!\}\{2^{r-1}r!\}4 = (2r)!/r!r!2 \geq 3 .$$

If $r < s$, then

$$\chi(M/Z) = 2^{r+s-1}(r+s)!/\{2^{r-1}r!\}\{2^{s-1}s!\}2 = (r+s)!/r!s! > (2r)!/r!r! \geq 2.$$

(7) $n > 1$ because M is not a sphere $\mathbf{S}^2$. If $n = 2$ then $M = \mathbf{Sp}(2)/\mathbf{U}(2) = \mathbf{SO}(5)/\mathbf{SO}(2) \times \mathbf{SO}(3)$ was considered under (3). Now suppose $n > 2$; then $\chi(M/Z) = 2^n n!/n!2 = 2^{n-1} > 2$.

(8) $\chi(M/Z) = 2^{2n}(2n)!/\{2^n n!\}\{2^n n!\}2 = (2n)!/n!n!2$ which is >2 for $n > 1$; and we exclude the case $n = 1$ because M is not a sphere $\mathbf{S}^4$.

(9) $\chi(M/Z) = 2^{10}\cdot 3^4\cdot 5\cdot 7/8!\cdot 2 = 36 > 2$.

(10) $\chi(M/Z) = 2^{10}\cdot 3^4\cdot 5\cdot 7/2^8\cdot 3^4\cdot 5 = 28 > 2$.

Hence our assertions are proved in case rank G=rank K. Now we must check the spaces listed under (2) and (6). For those spaces $M = G/K$ we will decompose $\mathbf{I}(M)$ as a union of components $\alpha_i G$, $\alpha_1 = 1$, such that its isotropy subgroup is a union of components $\alpha_i K$. If $z \in Z$, say $z \in \alpha_i G$, then z and α_i have the same action on $H^*(M, \mathbf{R})$, the space of $ad_G(K)$-invariants on $\Lambda^*\mathbf{P}'$. Thus we must analyse the action of K on $\mathbf{P}'$, picking out an invariant $\varphi \in \Lambda^k\mathbf{P}'$, such that $0 < k < \dim M$ and such that $\alpha_i(\varphi) = \varphi$ whenever Z meets $\alpha_i G$.

(2) $M = \mathbf{SU}(2n)/\mathbf{SO}(2n)$. Here $G = \mathbf{SU}(2n)/\{\pm\mathbf{I}\}$ has center $\mathbf{Z}_n$ which has index 2 in $Z \cong \mathbf{Z}_{2n}$. Note that $n > 1$ because M is not a sphere $\mathbf{S}^2$. We have [3, p. 88] $\mathbf{I}(M) = G \cup sG \cup \alpha G \cup s\alpha G$ with isotropy subgroup $K \cup sK \cup \alpha K \cup s\alpha K$ where s is the symmetry and $ad(\alpha)|_K = ad(a)|_K$ for a matrix $a = \mathrm{diag}\{-1; 1, \cdots, 1\} \in \mathbf{O}(2n)$. If $\mathbf{Z}_2$ denotes the subgroup of order 2 in Z, then $\mathbf{Z}_2 = \{1, \beta\}$ with $\beta \in \alpha G$. Thus we need only find a nonzero K-invariant $\varphi \in \Lambda^k\mathbf{P}'$, $0 < k < \dim M$, such that $\alpha(\varphi) = \varphi$.

The action of $K = \mathbf{SO}(2n)$ on the second symmetric power $S^2(\mathbf{R}^{2n})$ decomposes as $\phi \oplus \pi$, where ϕ is the (trivial) representation on the span of the element representing the invariant inner product on $\mathbf{R}^{2n}$, and π is equivalent to the representation of K on $\mathbf{P}'$.

Let $\omega \in \Lambda^{2n^2+n-1}(\mathbf{P}')$ denote the volume element of M. We check that $\alpha(\omega) = -\omega$, i.e. that α has determinant -1 on $\mathbf{P}'$. For if the matrix a of α has form $\mathrm{diag}\{-1; 1, \cdots, 1\}$ in a basis $\{v_1, \cdots, v_{2n}\}$ of $\mathbf{R}^{2n}$, then its (-1)-eigenvectors on $S^2(\mathbf{R}^{2n})$ are the $v_1\cdot v_i$, $2 \leq i \leq 2n$, which are odd in number.

Borel [1] has shown that the real cohomology of M is that of $\{\mathbf{S}^5 \times \mathbf{S}^9 \times \cdots \times \mathbf{S}^{4n-3}\} \times \mathbf{S}^{2n}$. First let $n = 2$. Then the product is $\mathbf{S}^4 \times \mathbf{S}^5$ so that $H^*(M; \mathbf{R})$ has basis $\{1, \varphi_4, \varphi_5, \omega\}$, where $\varphi_i \in H^i(M; \mathbf{R})$ and $\varphi_4 \wedge \varphi_5 = \omega$. Furthermore

$\alpha(\omega) = -\omega$ and $\chi(M/\mathbf{Z}_2) = \dfrac{1}{2}\chi(M) = 0$ imply $\alpha(\varphi_4) = -\varphi_4$ and $\alpha(\varphi_5) = \varphi_5$. *Thus $M/\mathbf{Z}_2$ is a real cohomology 5-sphere of dimension* 9. Now let $n > 2$, so that $H^*(M; \mathbf{R})$ is generated by elements $\varphi_i \in H^i(M; \mathbf{R})$ of degrees $5, 9, \cdots, 4n-3$, and $2n$ such that $(\varphi_5 \wedge \varphi_9 \wedge \cdots \wedge \varphi_{4n-3}) \wedge \varphi_{2n} = \omega$. If $\alpha(\varphi_i) = \varphi_i$ and $\alpha(\varphi_j) = \varphi_j$ for two distinct indices i, j, then M/Z is not a real cohomology sphere of any sort. If $\alpha(\varphi_i) = \varphi_i$ for a unique index i, then $\alpha(\varphi_j) = -\varphi_j$ for $j \neq i$. There are two indices $j \neq k$ distinct from i because $n \geq 3$, and now α preserves both φ_i and $\varphi_j \wedge \varphi_k$, so again M/Z is not a real cohomology sphere of any sort.

(6) $M = \mathbf{SO}(4r+2)/\mathbf{SO}(2r+1) \times \mathbf{SO}(2r+1)$, grassmannian of oriented $(2r+1)$-planes in an oriented $\mathbf{R}^{4r+2}$. Then $Z = \{1, \beta, \beta^2, \beta^3\} \cong \mathbf{Z}_4$, where β is orthocomplementation, and $\beta^2 = -I$ reverses orientation of $(2r+1)$-planes. We have $K = K_1 \times K_2$ with $K_i \cong \mathbf{SO}(2r+1)$. Let αG denote the component of $\mathbf{I}(M)$ containing β. Then $ad(\alpha)$ has order 2 and interchanges K_1 with K_2. Viewing $\mathbf{G}$ as the space of antisymmetric real matrices of degree $4r+2$, we identify an element of $\mathbf{P}$ with its upper right hand block of degree $2r+1$, and then $K = K_1 \times K_2$ acts on $\mathbf{P}$ by $(k_1, k_2)\colon \mathbf{A} \to k_1 A k_2^{-1}$. Now α acts on $\mathbf{P}$ by $A \to {}^tA$ transpose, so the multiplicity of its (-1)-eigenvalue there is $(2r+1)(2r)/2 = 2r^2 + r$. Thus α acts on the volume element ω by: $\alpha(\omega) = \omega$ if r is even, $\alpha(\omega) = -\omega$ if r is odd.

If $r = 1$, then $M = \mathbf{SO}(6)/\mathbf{SO}(3) \times \mathbf{SO}(3) = \mathbf{SU}(4)/\mathbf{SO}(4)$, and, as seen above, the 9-dimensional manifold M/Z is a real cohomology 5-sphere. Now suppose $r \geq 2$, so that $\dim M \geq 25$. Then the inclusion of M into the grassmannian of oriented $(2r+1)$-planes in $\mathbf{R}^\infty$ is an isomorphism on cohomology of degrees 4 and 8, so the Pontrjagin classes p_1 and p_2 of M are nonzero. Recall $p_i = (-1)^i c_{2i}(\tau_C)$, and $c_{2i}(\eta) = c_{2i}(\bar{\eta})$ for any complex vector bundle η, where c_j is the j-th Chern class, and τ is the tangent bundle. As $\alpha(\tau_C)$ is τ_C or $\bar{\tau}_C$, now $\alpha(p_1) = p_1$ and $\alpha(p_2) = p_2$. Thus M/Z is not a real cohomology sphere.

3. Products of even spheres

We now work out the last ingredient of our main result, proving

3.1. Proposition. *Let $\overline{M} = \mathbf{S}^{2r_1} \times \cdots \times \mathbf{S}^{2r_n}$, product of $m \geq 1$ even dimensional spheres, and $\Gamma \subset \mathbf{I}(\overline{M})$ be a finite subgroup such that $M = \overline{M}/\Gamma$ is a riemannian symmetric space.*

1. *$H^*(M; \mathbf{R}) = H^0(M; \mathbf{R})$ if and only if Γ consists of all $\gamma = \gamma_1 \times \cdots \times \gamma_m$, where γ_i is either the identity map or the antipodal map of the i-th factor $\mathbf{S}^{2r_i}$ of $\overline{M}$.*

2. *M is a real cohomology* (dim M)*-sphere if and only if Γ consists of all $\gamma = \gamma_1 \times \cdots \times \gamma_m$ as above such that the number of γ_i which are antipodal maps, is even.*

Proof. Let $\nu_i \in \mathbf{I}(\overline{M})$ act on the factors of $\overline{M}$ by the identity on $\mathbf{S}^{2r_s}$ for

$i \neq s$, and by the antipodal map on $\mathbf{S}^{2r_i}$. Let Δ denote the group generated by the ν_i, Δ' the subgroup of index 2 consisting of products of an even number of ν_i, and θ_i denote the character on Δ such that $\theta_i(\nu_s) = 1$ for $i \neq s$, and $\theta_i(\nu_i) = -1$. Then the 2^m characters $\theta_{i_1}\theta_{i_2} \cdots \theta_{i_s}$, $1 \leq i_1 < \cdots < i_s \leq m$, are all the characters of Δ, and $\theta_1\theta_2 \cdots \theta_m$ is the only nontrivial one which annihilates Δ'.

Let $\omega_i \in H^*(\overline{M}; \mathbf{R})$ be the $\mathbf{I}_0(\overline{M})$-invariant differential form of degree $2r_i$, which annihilates the tangent space to the factors $\mathbf{S}^{2r_s}$, $s \neq i$, of $\overline{M}$, and restricts to the volume element of $\mathbf{S}^{2r_i}$. Then $\nu_i^*\omega_s = \theta_s(\nu_i)\cdot\omega_s$. If $\delta \in \Delta$, then δ acts on $\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}$ by scalar multiplication with $(\theta_{i_1} \cdots \theta_{i_s})(\delta)$. But the 2^m elements $\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}$, $1 \leq i_1 < \cdots < i_s \leq m$, are a basis of of $H^*(\overline{M}; \mathbf{R})$. Hence

3.2. Lemma. *If $\Psi \subset \Delta$, then the Ψ-invariants on $H^*(\overline{M}; \mathbf{R})$ are just the span of* the $\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}$ *such that $\theta_{i_1} \cdots \theta_{i_s}$ annihilates Ψ.*

Now let Γ be a subgroup of Δ, i.e. suppose that $\overline{M}/\Gamma$ is symmetric. Then $H^*(\overline{M}/\Gamma; \mathbf{R}) = H^0(\overline{M}/\Gamma; \mathbf{R})$ if and only if none of the $\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}$ are Γ-invariant for $s > 0$. By Lemma 3.2 this latter condition is that no nontrivial character on Δ can annihilate Γ, i.e. Δ/Γ has no nontrivial character, i.e. $\Gamma = \Delta$. Thus the first assertion of the proposition is proved. $\overline{M}/\Gamma$ is a real cohomology sphere if and only if 1 and $\omega_1 \wedge \cdots \wedge \omega_m$ generate the Γ-invariants on $H^*(\overline{M}; \mathbf{R})$. Lemma 3.2 formulates the latter as the condition that $\theta_1\theta_2 \cdots \theta_m$ is the only nontrivial character on Δ, which annihilates Γ, i.e. that $\Gamma = \Delta'$. Thus the second assertion of the proposition is proved.

4. Proof of Theorem 1

We prove Theorem 1, stated at the beginning of this note.

M is a compact connected riemannian symmetric space, and $M = \overline{M}/\Gamma$ as in the notation of § 1.

If $\overline{M}$ is an odd sphere $\mathbf{S}^{2n-1}$, then $Z = \{\pm I\} \subset G = \mathbf{SO}(2n)$ acts trivially on the real cohomology of $\overline{M}$; so $\overline{M}$ and its associated projective space $\overline{M}/Z = \mathbf{S}^{2n-1}/\{\pm I\}$ are real cohomology spheres. If $\overline{M}$ is a product of even spheres, and Γ is the group described in case (2) of the theorem, then $\overline{M}/\Gamma$ is a real cohomology sphere by Proposition 3.1. If $\overline{M}$ is $\mathbf{SU}(3)/\mathbf{SO}(3)$, then $\dim \overline{M} = 5$, and

$$H^1(\overline{M}; \mathbf{R}) = 0, \text{ because } \overline{M} \text{ is simply connected,}$$

$$H^2(\overline{M}; \mathbf{R}) = 0, \text{ because } \overline{M} \text{ is not hermitian symmetric,}$$

$$H^3(\overline{M}; \mathbf{R}) = H^4(\overline{M}; \mathbf{R}) = 0 \text{ by Poincaré duality,}$$

so $\overline{M}$ is a real cohomology sphere; further $Z = \mathbf{Z}_3$, center of $G = \mathbf{SU}(3)$, so $\overline{M}/Z = \{\mathbf{SU}(3)/\mathbf{Z}_3\}/\mathbf{SO}(3)$ is a real cohomology sphere. Finally if $\overline{M} = \mathbf{SO}(5)/\mathbf{SO}(2) \times \mathbf{SO}(3)$ (oriented real grassmannian), then $\overline{M}/Z$

$= \overline{M}/\{\pm I\} = \mathbf{O}(5)/\mathbf{O}(2) \times \mathbf{O}(3)$ (nonoriented real grassmannian) is a real cohomology sphere by Proposition 2.1. Thus the spaces M listed in Theorem 1 are real cohomology (dim M)-spheres.

Conversely, let M be a real cohomology (dim M)-sphere. We run through the alternatives of Lemma 1.2. If M is a circle, it is an odd sphere, listed under (1) in Theorem 1. If M is irreducible, then it is a sphere, $\mathbf{SU}(3)/\mathbf{SO}(3)$, or $\mathbf{SO}(5)/\mathbf{SO}(2) \times \mathbf{SO}(3)$, by Proposition 2.1, and then M is a sphere or real projective space, $\mathbf{SU}(3)/\mathbf{SO}(3)$ or $\{\mathbf{SU}(3)/\mathbf{Z}_3\}/\mathbf{SO}(3)$, or $\mathbf{SO}(5)/\mathbf{SO}(2) \times \mathbf{SO}(3)$ or $\mathbf{O}(5)/\mathbf{O}(2) \times \mathbf{O}(3)$; even projective spaces are eliminated both by nonorientability and by $\chi = 1$, and $\mathbf{SO}(5)/\mathbf{SO}(2) \times \mathbf{SO}(3)$ is eliminated by $\chi = 4$; thus M is listed under (1), (2), (3) or (4) of Theorem 1. If $\overline{M}$ is reducible, then it is a product of even dimensional spheres by Lemma 1.2 and Proposition 2.1, and then M is listed under (2) of Theorem 1, by Proposition 3.1.

5. Extension to Theorem 2

We modify the proof of Theorem 1 in such a way as to obtain Theorem 2.

Let M be a connected riemannian symmetric space. Then we have the universal riemannian covering $\varphi: N \to M = N/\varDelta$, and decompose $N = N_0 \times N' \times N''$, where N_0 is a euclidean space, N' a product of compact simply connected irreducible symmetric spaces, and N'' a product of noncompact irreducible symmetric spaces. $\varDelta$ has trivial projection on $\mathbf{I}(N'')$, so $\varDelta \subset \mathbf{I}(N_0) \times \mathbf{I}(N')$; $\varDelta$ has finite projection on $\mathbf{I}(N')$, so $\varDelta_0 = \varDelta \cap \mathbf{I}(N_0)$ is a subgroup of finite index; in particular $\varDelta_0$ has finite index in the projection of $\varDelta$ to $\mathbf{I}(N_0)$. The projection of $\varDelta$ to $\mathbf{I}(N_0)$ is a group of euclidean translations, and this decomposes $N_0 = N_0' \times N_0''$, where $\varDelta$ acts trivially on N_0'', and N_0' has compact quotient by the projection of $\varDelta$ to $\mathbf{I}(N_0)$. Now define

$$\overline{M} = \overline{M}' \times \overline{M}'', \quad \overline{M}' = (N_0' \times N')/\varDelta_0, \quad \overline{M}'' = N_0'' \times N'' ,$$

so that

$$M = M' \times M'', \quad \text{where} \quad M' = \overline{M}'/\Gamma, \quad M'' = \overline{M}'', \quad \Gamma = \varDelta/\varDelta_0 ,$$

and $\varphi: N \to M$ factors through the covering $\pi: \overline{M} \to M = \overline{M}/\Gamma$. M' is a compact connected riemannian symmetric space; M'' is contractible because it is the product of a euclidean space N_0'' and a product N'' of noncompact irreducible symmetric spaces; under the inclusion $\iota: M' \to M$, now $\iota^*: H^*(M; A) \cong H^*(M'; A)$ for any coefficient ring A. This reduces the proof of Theorem 2 to the case dim $M'' = 0$ where M is compact.

Now let M be a compact connected riemannian symmetric space which is a real cohomology n-sphere, where $0 \leq n \leq \dim M$. *Recall our convention that a* 0-*sphere means a single point.* As in § 1 we decompose $M = \overline{M}/\Gamma$, $\overline{M} = M_0 \times M_1 \times \cdots \times M_r$, where M_0 is a flat riemannian torus and the other M_i are compact simply connected irreducible symmetric spaces. Lemma 1.1 holds but Lemma 1.2 must be modified.

If $\dim M_0 > 0$, then, as before, $n = 1$ and M_0 is a circle. For $i > 0$, now M_i contributes nothing to $H^*(M; \mathbf{R})$, so M_i is an even dimensional sphere $\mathbf{S}^{2r_i}$ by Proposition 2.1. Let Γ' denote the projection of Γ to $\mathbf{I}(M_1 \times \cdots \times M_r)$. Then $\Gamma \to \Gamma'$ is an isomorphism by construction of M_0, and $\Gamma' \cong (\mathbf{Z}_2)^r$ consisting of all $\gamma' = \gamma_1 \times \cdots \times \gamma_r$ where γ_i is 1 or the antipodal map on $M_i = \mathbf{S}^{2r_i}$, by Proposition 3.1. Thus Γ consists of all $\gamma = \gamma_0 \times \gamma'$, where $\gamma' \in \Gamma'$ as just described, and $\gamma_0 = \theta(\gamma')$ for some arbitrary fixed character θ on Γ'. Since there are 2^r choices of θ, our assertions of Theorem 2 are now proved for the case $\dim M_0 > 0$.

Now we assume $\dim M_0 = 0$, so $M = M_1 \times \cdots \times M_r$.

Suppose that $\overline{M}/Z$ is a real cohomology 0-sphere, i.e. that $H^*(\overline{M}/Z; \mathbf{R}) = H^0(\overline{M}/Z; \mathbf{R})$. Then Proposition 2.1 tells us that $M_i = \mathbf{S}^{2r_i}$ even sphere. If $n = 0$, then Proposition 3.1 says $\Gamma = Z$. If $n > 0$, then Lemma 3.2 says that $H^*(M; \mathbf{R})$ is spanned by 1 and by some $\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}$, where ω_i is the volume element of M_i, $1 \leq i_1 < \cdots < i_s \leq r$, $s > 0$, $n = 2r_{i_1} + \cdots + 2r_{i_s}$, and $\Gamma \cong (\mathbf{Z}_2)^{r-1}$ is the kernel of the character $\theta_{i_1} \cdots \theta_{i_s}$. Thus there are $2^r - 1$ possibilities for Γ, and the assertions of Theorem 2 is proved for the case where M/Z is a real cohomology 0-sphere.

Now we assume that M/Z is not a real cohomology 0-sphere. Then $n > 0$, and M/Z is a real cohomology n-sphere. We re-order the M_i now, so that M_1/Z_1 is a real cohomology n-sphere and the other M_i/Z_i are real cohomology 0-spheres. Proposition 2.1 tells us

(i) if $i > 1$, then M_1 is an even dimensional sphere;

(ii) if $n = \dim M_1$, then M_1 is an odd sphere, is $\mathbf{SU}(3)/\mathbf{SO}(3)$ or is $\mathbf{SO}(5)/\mathbf{SO}(2) \times \mathbf{SO}(3)$.

5.1. Lemma. *Let M_1 be a compact simply connected irreducible riemannian symmetric space, and Z_1 the centralizer of $\mathbf{I}_0(M_1)$ in $\mathbf{I}(M_1)$, and suppose that M_1/Z_1 is a real cohomology n-sphere where $0 < n < \dim M_1$. Then $n = 5$, $\dim M_1 = 9$ and $M_1 = \mathbf{SU}(4)/\mathbf{SO}(4) = \mathbf{SO}(6)/\mathbf{SO}(3) \times \mathbf{SO}(3)$.*

Proof. Let $m = \dim M_1$. Then $H^m(M_1/Z_1; \mathbf{R}) = 0$ says that Z_1 acts nontrivially on $H^*(M_1; \mathbf{R})$, so M_1 is one of the ten (types of) spaces listed at the beginning of the proof of Proposition 2.1.

If $\chi(M_1) \neq 0$, then $H^k(M_1; \mathbf{R}) = 0$ for k odd, so $H^k(M_1/Z_1; \mathbf{R}) = 0$ for k odd; thus n is even and $\chi(M_1/Z_1) = 2$. Following the proof of Proposition 2.1 for that case, we see $M_1 = \mathbf{SO}(5)/\mathbf{SO}(2) \times \mathbf{SO}(3)$, so $n = m = 6$, contradicting $n < m$. Thus $\chi(M_1) = 0$. Following the proof of Proposition 2.1 for that case we see that M_1 is the 9-dimensional $\mathbf{SU}(4)/\mathbf{SO}(4) = \mathbf{SO}(6)/\mathbf{SO}(3) \times \mathbf{SO}(3)$ with $Z_1 \cong \mathbf{Z}_4$ and $n = 5$. q.e.d.

Returning to the proof of Theorem 2, let $t = r - 1$; then we need only examine the cases

(1) $\overline{M} = \mathbf{S}^{2m+1} \times \mathbf{S}^{2r_1} \times \cdots \times \mathbf{S}^{2r_t}$, $m > 0$, $t \geq 0$;

(2) $\overline{M} = \{\mathbf{SU}(3)/\mathbf{SO}(3)\} \times \mathbf{S}^{2r_1} \times \cdots \times \mathbf{S}^{2r_t}$;
(3) $\overline{M} = \{\mathbf{SO}(5)/\mathbf{SO}(2) \times \mathbf{SO}(3)\} \times \mathbf{S}^{2r_1} \times \cdots \times \mathbf{S}^{2r_t}$;
(4) $\overline{M} = \{\mathbf{SO}(6)/\mathbf{SO}(3) \times \mathbf{SO}(3)\} \times \mathbf{S}^{2r_1} \times \cdots \times \mathbf{S}^{2r_t}$.

In each case let Γ' be the projection of Γ to $\mathbf{I}(\mathbf{S}^{2r_1} \times \cdots \times \mathbf{S}^{2r_t})$. Then Proposition 3.1 says that $\Gamma' \cong (\mathbf{Z}_2)^t$ consists of all $\gamma' = \gamma_1 \times \cdots \times \gamma_t$ where γ_i is 1 or the antipodal map on $\mathbf{S}^{2r_i}$. And in each case let $\Gamma^0 = \Gamma \cap \mathbf{I}(M_1)$, kernel of $\Gamma \to \Gamma'$.

In cases (1) and (2), where Z_1 acts trivially on $H^*(M_1; \mathbf{R})$, the symmetric space $\overline{M}/\Psi$ is a real cohomology (dim M_1)-sphere if and only if Ψ projects onto $\Gamma' = Z_2 \times Z_3 \times \cdots \times Z_{t+1} \cong (\mathbf{Z}_2)^t$. For the action of $\gamma = \gamma^0 \times \gamma' \in Z = Z_1 \times \Gamma'$ on real cohomology of $\overline{M}$ is just that of $1 \times \gamma'$. In case (1) this means that Γ can be $Z \cong (Z_2)^{t+1}$ if $\Gamma^0 \neq \{1\}$; if $\Gamma^0 = \{1\}$ then Γ can be any of the 2^t groups

$$\Gamma_\theta = \{\theta(\gamma') \times \gamma' : \gamma' \in \Gamma'\} \cong (\mathbf{Z}_2)^t ,$$

where θ is a character on Γ'. In case (2) it means either that $\Gamma^0 \neq \{1\}$ and $\Gamma = Z \cong \mathbf{Z}_3 \times (\mathbf{Z}_2)^t$, or that $\Gamma^0 = \{1\}$ and $\Gamma = \Gamma' \cong (\mathbf{Z}_2)^t$.

In cases (3) and (4), where M_1 is not a real cohomology sphere because of a nonzero element $\omega_0 \in H^4(M_1; \mathbf{R})$, that element ω_0 is sent to its negative by a generator z_0 of Z_1. Let ω_i denote the volume element of $\mathbf{S}^{2r_i}$; now we require that no form $\omega_{i_1} \wedge \cdots \wedge \omega_{i_s} \neq 1$, $0 \leq i_1 < \cdots < i_s \leq t$, can be Γ-invariant. As for Proposition 3.1, it follows that Γ separately contains the generator of each Z_i. Thus $\Gamma = Z$, so $\Gamma \cong (\mathbf{Z}_2)^{t+1}$ in case (3) and $\Gamma \cong \mathbf{Z}_4 \times (\mathbf{Z}_2)^t$ in case (4). Conversely, $\Gamma = Z$ implies $M = (M_1/Z_1) \times (\mathbf{S}^{2r_1}/\mathbf{Z}_2) \times \cdots \times (\mathbf{S}^{2r_t}/\mathbf{Z}_2)$, $\mathbf{R}$-cohomologically equivalent to the real cohomology sphere M_1/Z_1. Hence the proof of Theorem 2 is complete.

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